Reply by Authors to J. G. Simmonds

ALLAN B. PIFKO* AND MARTIN A. GOLDBERG†
Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

ADDITIONAL evidence of the accuracy and convergence associated with a recently developed iterative technique[1, 2] was presented in Ref. 3. This was accomplished by comparing the iterative solution for a uniformly loaded annular membrane, fixed at the outer boundary and free of tractions and support at the inner boundary with a power series solution.

In the preceding comment Simmonds draws attention to that portion of the paper dealing with the power series solution. The alternate procedure employed by Simmonds leads precisely to the solution obtained by the authors. Indeed, making use of the transformation

$$\Phi = (1 - \lambda^2)/a_1^3$$
$$\eta = 2 - (1 + \nu)(1 - \lambda^2) \tag{1}$$

in the controlling algebraic equation for a_1, λ, and ν [Eqs. (24b), Ref. 4] leads directly to the relations

$$1.3a_1^3 = (1 + \nu)A_1^3$$
$$(1 - \lambda^2)/a_1^3 = (1 - \Lambda^2)/A_1^3 \tag{2}$$

given by Simmonds. Conversely, the governing equation for Φ [Eq. (10) of the preceding comment] can be recast to permit a direct solution for a priori specified values of ν and λ.

Invariant relations such as Eqs. (2) increase the usefulness of the available numerical results with a minimal amount of additional calculations. However, Simmonds remarks are somewhat optimistic. The results presented in Table 1 of Ref. 3 are insufficient to provide a complete spectrum of coefficients. To take full advantage of the invariant relations requires a complete table of results based on $\nu = 0.5$. Therefore, to augment the results given in Ref. 3, a more complete Table 1 (for $\nu = 0.5$) is given. The necessary equations to be used in conjunction with this table are

$$\Lambda = \left[1 - \frac{1.5}{1 + \nu}(1 - \lambda^2)\right]^{1/2} \tag{3}$$
$$A_1 = \left(\frac{1.5}{1 + \nu}\right)^{1/3} a_1$$

which replace Eqs. (19) of the preceding note.

Table 1 Coefficients a_1 for $\nu = 0.5$ and various values of λ

λ	a_1	λ	a_1
0	1.845	0.36	1.649
0.01	1.845	0.37	1.639
0.02	1.844	0.38	1.630
0.03	1.844	0.39	1.620
0.04	1.842	0.40	1.610
0.05	1.841	0.41	1.600
0.06	1.839	0.42	1.590
0.07	1.836	0.43	1.580
0.08	1.834	0.44	1.570
0.09	1.831	0.45	1.559
0.10	1.827	0.46	1.548
0.11	1.824	0.47	1.538
0.12	1.820	0.48	1.527
0.13	1.816	0.49	1.516
0.14	1.811	0.50	1.505
0.15	1.806	0.51	1.494
0.16	1.801	0.52	1.483
0.17	1.795	0.53	1.471
0.18	1.790	0.54	1.460
0.19	1.784	0.55	1.448
0.20	1.777	0.56	1.436
0.21	1.771	0.57	1.424
0.22	1.764	0.58	1.412
0.23	1.757	0.59	1.400
0.24	1.750	0.60	1.388
0.25	1.743	0.61	1.375
0.26	1.735	0.62	1.362
0.27	1.727	0.63	1.350
0.28	1.719	0.64	1.336
0.29	1.711	0.65	1.323
0.30	1.703	0.66	1.310
0.31	1.694	0.67	1.296
0.32	1.685	0.68	1.282
0.33	1.676	0.69	1.268
0.34	1.667	0.70	1.254
0.35	1.658		

Received September 17, 1964.

* Research Engineer, Research Department. Associate Member AIAA.

† Formerly Head of Applied Mechanics, Research Department; now Associate Professor, Aerospace Engineering and Applied Mechanics Department, Polytechnic Institute of Brooklyn, Brooklyn, N. Y.

References

[1] Goldberg, M. A., "A modified large deflection theory of plates," *Proceedings of the Fourth U. S. National Congress of Applied Mechanics* (American Society of Mechanical Engineers, New York, 1962), pp. 611–618.

[2] Goldberg, M. A. and Pifko, A. B., "Large deflection analysis of uniformly loaded annular membranes," AIAA J. **1**, 2111–2115 (1963).

[3] Pifko, A. B. and Goldberg, M. A., "Iterative and power series solutions for the large deflection of an annular membrane," AIAA J. **2**, 1340–1342 (July 1964).

[4] Pifko, A. B. and Goldberg, M. A., "Comments on an iteration procedure for the large deflection analysis of initially flat membranes," Grumman Research Dept. Publ. RM-232 (June 1964).

Distribution of Nearly Circular Orbits

DAVID A. CONRAD*
Hughes Aircraft Company, El Segundo, Calif.

1. Introduction

IN a recent paper[1] the theory of errors was employed to assess the precision of a guidance system designed to place a payload in a prescribed circular orbit. The essence of the analysis consists in determining the distribution of the square of the orbital eccentricity

$$e^2 = f(\beta, r, v) = \sin^2\beta + [(rv^2/gR^2) - 1]^2 \cos^2\beta \tag{1}$$

based on assumed normal distributions of the orbital parameters β, r, v at burnout. The distance r is measured from the center of the earth, v is the velocity, β is the heading angle measured outward from the normal to the radius vector, and g and R are the gravitational constant and earth radius.

In the referenced paper, the solution is obtained, approximately, first in terms of the gamma distribution and then,

Received June 24, 1964.

* Senior Staff Engineer, Surveyor System Analysis Laboratory, Space Systems Division. Member AIAA.

even more approxmately, in terms of the normal distribution. It is the purpose of this comment to point out that Monte-Carlo methods can be easily used to compute directly the distribution to any desired degree of accuracy. The advantage of such an approach is in requiring a minimum of statistical theory to produce a precise result in an extremely short time.

2. Error Sources

As in the reference, it will be assumed that the parameters β, r, v are normally distributed with covariance matrix Σ. This matrix is obtained by a straightforward linear transformation of the covariance matrix of the fundamental error sources. Although it is convenient if this matrix is diagonal (independent error sources), this is by no means essential.

In the Monte-Carlo procedure it is necessary to generate (on a digital computer) random samples of β, r, v according to the desired normal distribution. This can be done either by generating a random vector of the fundamental error sources (possibly correlated) and then applying the linear transformation, or by directly generating the correlated samples of β, r, v. In both cases it is necessary to generate samples of correlated, normally distributed random variables.

3. Generation of Samples of Correlated Gaussian Variables†

There are well established computer methods for the generation of a vector x of independent Gaussian variables with zero mean and unit variance. It is desired to obtain the vector y with covariance matrix Σ_y by a linear transformation C on x:

$$y = Cx + \bar{y} \tag{2}$$

where $\bar{y}$ is the mean of y and

$$\Sigma_y = E[(y - \bar{y})(y - \bar{y})^T] = CC^T \tag{3}$$

where T indicates the transpose. If C is chosen as a lower triangular matrix, the elements $c_{ij}(j \leqq i)$ may be computed sequentially from the relations

$$\sum_{k=1}^{j} c_{ik}c_{jk} = \sigma_{ij} \qquad k \leq j \leq i \tag{4}$$

for $j = 1$, Eqs. (4) have only one term yielding

$$c_{11} = \sigma_{11}^{1/2} \qquad c_{k1} = \sigma_{k1}/c_{11} \tag{5}$$

For $j = 2$, there are two terms, but only one new unknown element is involved. Hence,

$$c_{22} = (\sigma_{22} - c_{21}^2)^{1/2} \qquad c_{k2} = (\sigma_{k2} - c_{k1}c_{21})/c_{22} \tag{6}$$

The pattern continues for higher j, with only one new element appearing in each equation.

With the elements of C known, the correlated components of the random vector y are computed directly in terms of the independent components of x from (2).

4. The Monte-Carlo Simulation

The calculation, then, consists in generating samples of the normally distributed independent components of x (in this case a three-vector), computing the components β, r, v of y from (2), and then computing e^2 from (1).

The time required for such a computation on an IBM 7090 would be less than 1 min for 10,000 runs, including the sorting of the results necessary to produce the complete distribution. An experienced programmer should be able to write and check out such a program in about one week.

In comparison, the method proposed by Beckwith[1] requires the computation of the eigenvalues of a large ($n = 20$ to 30) matrix, or else the assumption that e^2 is normal. In both cases the accuracy of the approximations is difficult to evaluate and requires considerably more statistical sophistication than the direct approach afforded by the Monte-Carlo method.

It should be noted that Skidmore and Braham[2] have presented an alternate approach to the same problem, namely numerical integration of the trivariate Gaussian distribution of β, r, v over the region in β, r, v space bounded by a particular value of e^2. This procedure is not difficult to apply in the event that a multivariate Gaussian integration program is available. This latter reference also contains a relatively complete description of the general problem of statistical analysis of satellite trajectories

References

[1] Beckwith, R. E., "Approximate distribution of nearly circular orbits," AIAA J. 2, 913–916 (1964).

[2] Braham, H. S. and Skidmore, L. J., "Guidance error analysis of satellite trajectories," J. Aerospace Sci. 29, 1091–1101 (1962).

† The methods of this section were pointed out to the author by L. J. Skidmore.

Reply by Author to D. A. Conrad

R. E. BECKWITH*
University of Southern California, Los Angeles, Calif.

1. Introduction

THE preceding comment advocates the superiority of Monte Carlo methods over the methods presented in Ref. 1 for answering certain questions pertaining to the precision of "nearly circular" orbits. Without agreeing or disagreeing with D. A. Conrad, the present note seeks to place the issue in clearer perspective.

2. Monte Carlo Method

A rich, if diffuse, literature exists in *model sampling* or *Monte Carlo method*, as the technique is now more popularly known. Principal applications have been made in the areas of nuclear studies, logistics, heuristic problem solving, and evaluation of complicated integrals. Whether the original problem is deterministic or stochastic, the Monte Carlo method consists of building and "playing" an appropriate game of chance in which stochastic convergence of relevant sample statistics to basic system parameters is assured under very general conditions by the strong law of large numbers and the central limit theorem and in which distribution laws of arbitrary statistics can be determined to any desired degree of accuracy by appropriate techniques, such as that due to Kolmogorov-Smirnov.

The efficient use of Monte Carlo techniques depends heavily upon one's ability to transform one game of chance into another one, in which the expected values of the important system statistics remain unchanged, but for which the statistics in the transformed game have smaller sampling variances than those of the original game.

The writer, who has used the Monte Carlo method to treat a great variety of problems which posed particular analytic difficulty, has no quarrel with others who would employ it for the same reason.† He would point out, however, that many questions that can be answered by a straightforward sensitivity analysis of an analytic model become costly exercises

Received July 21, 1964.

* Associate Professor, Graduate School of Business Administration.

† Today's rather widespread practice of usage of Monte Carlo or gaming techniques by persons who are unprepared either to recognize analytically solvable problems or treat them by analytic methods, or to comprehend the intricacies to modern statistical theory, a commentary not relevant to this reply to the preceding note, is to be deplored, however.